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ENERGY OF GRAVITATIONAL FIELD

POST–NEWTONIAN POLYTROPES IN ALTERNATIVE GRAVITATIONAL THEORY

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Relativistic corrections to the total energy of a compact object must be introduced in analisys of its stability in alternative gravitational theory. In the case of polytrope type of the equation of state in the form in the form of $P = K\rho^{1+1/n}$ all the Post-Newtonian corrections in GRT to gravitational energy involve integrals such as $I_{abcd} = \int \vartheta^{an+b}(\vartheta')^c \xi^d \, d\xi$ where n is the index of polytrope, a, b, c, d are integers and ϑ' is the derivative of Emden's function $\vartheta(\xi)$. In the present paper it is shown that all the considered integrals of the Post-Newtonian corrections are expressed through one - here we choosed I_{1202}. The analytical relations between other integrals and I_{1202} together with its tabled values are presented which allow, to get Post-Newtonian corrections to the gravitational energy for a wide range of alternative gravitational theories without complex calculations.

Contents

1. Introduction

The problem of stability of relativistic compact objects with different masses (neutron stars, supermassive objects in active galactical nucleus, etc.) has great importance for astrophysical applications. Relativistic corrections to the total energy of a compact object [1–4] must be taken into account in its analysis. In GRT it was done for polytropic models by W.A. Fowler [2]. In present paper we suggest an analytical method for determining stability in different gravitational theories.

All the Post-Newtonian corrections to gravitational energy involve integrals of the form:

$$I_{abcd} = \int \vartheta^{an+b}(\vartheta')^c \xi^d \, d\xi$$

where n is the polytrope index in the equation of state $P = K\rho^{1+1/n}$, a, b, c, d are integers and ϑ' is the derivative of Emden's function ϑ In the present paper it is shown that all the considered integrals of the Post-Newtonian corrections are expressed through one I_{1202} (has been chosen). The analytical expressions of other integrals over I_{1202} and its table values are presented giving Post-Newtonian corrections to gravitational energy of compact objects.

2. Classical polytropic energy

Analytical forms of polytropic models are discribed with the polytropic functions

$$f_{abcd} = \vartheta^{an+b}(\vartheta')^c\xi^d \tag{1}$$

and integrals over them

$$I_{abcd} = \int f_{abcd}\, d\xi. \tag{1$'$}$$

Using Leun–Emden's equation

$$\vartheta'' = -\frac{2}{\xi}\,\vartheta' - \vartheta^n \tag{2}$$

we get

$$f'_{abcd} = (an+b)f_{a,b-1,c+1,d} + (d-2c)I_{a,b,c,d-1} - c^* I_{a+1,b,c-1,d}. \tag{3}$$

Integrating this equation we come to:

$$(an+b)f_{a,b-1,c+1,d} + (d-2c)I_{a,b,c,d-1} - cI_{a+1,b,c-1,d} = f_{abcd}. \tag{3$'$}$$

The linear relation Eq. 3$'$ has been derived at by partial integration and using Leun–Emden's equation Eq. 2. This provides a method of calculating the integrals I_{abcd}.

Eqs. 3 and 3$'$ allow to get the analytical expression of energy radial dependence in classical case. It is $E_{\rm cl}$ defined as a sum of the thermal W and gravitational energy U

$$E_{\rm cl} = W + U = \int \varepsilon\, dm - \int \frac{Gm}{r}\, dm \tag{4}$$

where $\varepsilon = nP/\rho$ is the thermal energy per unit mass and m is the mass inside the sphere of the radius r. With the standard dimensionless variables ϑ and ξ defined as

$$\rho = \rho_c\vartheta^n, \quad r = \alpha\xi, \quad \alpha = \left[\frac{(n+1)K}{4\pi G}\,\rho^{1/n-1}\right]^{1/2} \tag{5}$$

we get

$$\begin{aligned} W &= \omega I_{1102} = \omega\int \vartheta^{n+1}\xi^2\, d\xi, \\ U &= \frac{n+1}{n}\,\omega I_{1013} = \frac{n+1}{n}\,\omega\int \vartheta^n\vartheta'\xi^3\, d\xi \end{aligned} \tag{6}$$

where $\omega = 4\pi\alpha^3 nP_c$, and P_c is the central pressure.

Integrals I_{1102} and I_{1013} as well as mass integral

$$I_{1002} = \int \vartheta^n\xi^2\, d\xi = -\vartheta'\xi^2 = f_{0012} \tag{7}$$

can be analytically derived $\{f_{abcd}\}$

$$\begin{aligned} I_{1102} &= \frac{1}{n-5}\left[(n+1)\xi^2\vartheta'(\xi\vartheta'+\vartheta) + 2\vartheta^{n+1}\xi^3\right] = \\ &= -\frac{1}{n-5}\left[(n+1)(f_{0023}+f_{0112}) + 2f_{1103}\right], \end{aligned} \tag{8}$$

$$I_{1013} = \frac{1}{n-5}\left[\vartheta^{n+1}\xi^3 + 3\xi^2\vartheta'(\xi\vartheta'+\vartheta)\right] = \frac{1}{n-5}\left[3f_{0023} + 3f_{0112} + f_{1103}\right]. \tag{9}$$

Applying standard integration methods this result can be heuristically received using linear relations Eqs. 3 and 3′. Substituting consequently the integrals I_{1102} and I_{1013} into each of the three terms of Eq. 3′ we obtain six equations; three of them form non-degenerate linear system for integrals I_{1102}, I_{1013} and I_{0022}

$$\begin{cases} I_{0022} - 2I_{1013} &= f_{0023}, \\ I_{0022} - I_{1102} &= f_{0112}, \\ (n+1)\, I_{1013} + 3I_{1102} &= f_{1103}. \end{cases}$$

Substituting Eqs. 6, 8 and 9 into Eq. 4 we obtain classical relation between polytropic energy and dimensionless radius ξ

$$E_{\rm cl}(\xi) = \frac{4\pi\alpha^3 P_c}{n-5}\left[\frac{1-n}{n}\,\vartheta^{n+1}\xi^3 + (n+1)(3-n)\xi^2\vartheta'(\xi\vartheta' + \vartheta)\right] \tag{10}$$

The known equation for total polytrope's energy ($\xi = \xi_1$) is

$$E_{\rm cl} = 4\pi\alpha^3 P_c\,\frac{(n+1)(3-n)}{n-5}\,\xi_1^3(\vartheta')^2. \tag{11}$$

3. Post–Newtonian energy of a polytrope

Polytropic models of stars can be presented as one-parameter sequence of equilibrium states with the same equation of state and different central densities ρ_c. The dependence of the total energy $E(M, \rho_c)$ on mass M and central density ρ_c is important for studing of stability in Post-Newtonian range. Putting E first derivative with respect to ρ_c to zero gives the relation between M and ρ_c for stationary configuration and the sign of the second derivative E by ρ_c indicates the stability of this state [3, 4].

The Post-Newtonian energy of a politrope can be defined as

$$E = E_{\rm cl} + \Delta W + \Delta U \tag{12}$$

where ΔW is the relativistic correction to the thermal energy caused by the deviation from polytropic equation of state and ΔU is the relativistic correction to Newtonian gravitational energy. The correction ΔW to thermal energy can be obtained [4] without using integral of the I_{abcd} type. At the same time, relativistic correction ΔU to gravitational energy of the Post-Newtonian polytrope depends on the choice of gravitational theory and is expressed through integral of the I_{abcd} type, as we show below in the case of GRT.

In GRT the Post-Newtonian correction to the gravitational energy is [3]

$$\Delta U_{GR} = I_{(1)} + I_{(2)} + I_{(3)} + I_{(4)} + I_{(5)} \tag{13}$$

where

$$I_1 = -\frac{G}{c^2}\int_0^M \varepsilon\,\frac{m\,dm}{r}, \tag{14}$$

$$I_2 = -\frac{1}{2}\frac{G^2}{c^2}\int_0^M \frac{m^2\,dm}{r^2}, \tag{15}$$

$$I_3 = -\frac{G}{c^2}\int_0^M \left(\int_0^m \varepsilon\,dm\right)\frac{dm}{r}, \tag{16}$$

$$I_4 = \frac{G^2}{c^2}\int\limits_0^M \left(\int\limits_0^m \frac{m\,dm}{r}\right)\frac{dm}{r}, \tag{17}$$

$$I_5 = -\frac{G^2}{c^2}\int\limits_0^M \left(\int\limits_0^r mr\,dr\right)\frac{m\,dm}{r^4}. \tag{18}$$

Note, that these integrals belong to the I_{abcd}-type. Eqs. 15 and 16 transform with dimensionless variables as

$$I_{(1)} = \frac{n}{n+1}\,\omega_1 I_{1113}, \tag{19}$$

$$I_{(2)} = -\frac{1}{2}\,\omega_1 I_{1024} \tag{20}$$

where

$$\omega_1 = \frac{G^2}{c^2}\frac{(4\pi)^{2/3}}{|\vartheta_1'\xi_1^2|^{7/3}}\,M^{7/3}\rho^{2/3} \tag{21}$$

and $\vartheta_1' = \vartheta'(\xi_1)$. Going to the standard dimensionless variables of Eq. 5 in Eqs. 16 and 17 the thermal and gravitational energy equations can be written as function of the radius ξ in Eqs. 6, 8 and 9

$$I_{(3)} = \frac{n}{n-5}\,\omega_1\left(I_{1024} + I_{1113} + \frac{2}{n+1}\,I_{2104}\right), \tag{22}$$

$$I_{(4)} = -\omega_1\left(\frac{1}{n-5}\,I_{1024} + 3I_{1024} + 3I_{1113}\right). \tag{23}$$

Eq. 18 leads to Eq. 14 integrating by parts ($Gm\,dm/r^4 = 4\pi\,dP$) then

$$I_{(5)} = \frac{1}{n}\,I_{(1)} = \frac{1}{n+1}\,\omega_1 I_{1113}. \tag{24}$$

Thus the definition of the Post Newtonian correction ΔU_{GR} leads to the evaluation of the three integrals I_{1113}, I_{1024}, I_{2104} of the I_{abcd}-type. A variety of integrating methods gives the expression for ΔU_{GR} with two integrals [3–6]. These integrals are $I_{(1)}(I_{1113})$ and $I_{(2)}(I_{1024})$ in [3, 4] or I_{1202} and I_{2104} in [5]. The method described above and based on Eqs. 3 and 3′ makes it possible to get all linear relations of the I_{abcd} type integrals and express ΔU as an integral. Eqs. 3 and 3′ together with integrals I_{0033}, I_{1202} and I_{0122} lead to five linear equations of the Eq. 3′-type with the right-hand sides are f_{1114}, f_{0034}, f_{1203}, f_{0123} and f_{0212}. Although it is impossible to obtain analytical form of the required integrals they $\{f_{abcd}\}$-type can be written through the integral (f. e. I_{1202})

$$I_{1113} = \frac{1}{n+2}\,\vartheta^{n+1}\xi^3 - \frac{3}{n+2}\,I_{1202}, \tag{25}$$

$$I_{1024} = -\frac{2}{3}\,\vartheta(\vartheta')^2\xi^3 - \frac{4}{3(n+2)}\,\vartheta^{n+2}\xi^3 - \frac{1}{3}\,\vartheta^2\vartheta'\xi^2 - \frac{1}{3}\,(\vartheta')^3\xi^4 - \frac{n-10}{3(n+2)}\,I_{1202}, \tag{26}$$

$$I_{2104} = \frac{2(1-2n)}{3(n+2)}\,\vartheta^{n+2}\xi^3 - \vartheta^{n+1}\vartheta'\xi\,\frac{2(n+1)}{3}\,\vartheta(\vartheta')^2\xi^3 - $$
$$-\frac{n+1}{3}\,\vartheta^2\vartheta'\xi^2 - \frac{n+1}{3}\,(\vartheta')^3\xi^4 - \frac{(n-8)(n-1)}{3(n+2)}\,I_{1202}. \tag{27}$$

Then the required integrals Eqs. 14–18 can be expressed with only I_{1202}

$$I_{(1)} = -\frac{3n}{(n+1)(n+2)}\,\omega_1 I_{1202}, \tag{28}$$

$$I_{(2)} = \frac{1}{6}\omega_1\left((\vartheta')^3\xi^4 - \frac{n-10}{n+2}I_{1202}\right), \quad (29)$$

$$I_{(3)} = \omega_1\left(\frac{n}{5-n}(\vartheta')^3\xi^4 - \frac{n(n-1)}{(n+1)(n+2)}I_{1202}\right), \quad (30)$$

$$I_{(4)} = \omega_1\left(-\frac{n+4}{3(5-n)}(\vartheta')^3\xi^4 + \frac{n-1}{3(n+2)}I_{1202}\right), \quad (31)$$

$$I_{(5)} = -\frac{3}{(n+1)(n+2)}\omega_1 I_{1202}. \quad (31)$$

Substituting Eqs. 28–32 to Eq. 13 and getting $\vartheta(\xi_1) = 0$ on the polytrope's surface the relativistic correction ΔU_{GR} to the gravitational energy in GRT is given by

$$\Delta U_{GR} = \frac{G^2}{c^2} k_{GR} M^{7/3}\rho^{2/3} \quad (33)$$

where

$$k_{GR} = -\frac{(4\pi)^{2/3}}{2|\vartheta_1'\xi_1^2|^{7/3}}\left[\frac{n+5}{n+1}I_{1202} + \frac{n-1}{n-5}(\vartheta')^3\xi^4\right]. \quad (34)$$

The following table containss numerical values of the I_{1202} for various indexes n.

Table 1.

n	ξ_1	$\int \vartheta^{n+2}\xi^2\, d\xi$	$-(\vartheta')^3\xi_1^4$	$-\vartheta'\xi_1^2$
0	$\sqrt{6}$	$16\sqrt{6}/35$	$8\sqrt{6}$	$2\sqrt{6}$
0.5	2.7527	1.03804	7.1772	3.7888
1	π	0.97026	π	π
1.5	3.6538	0.91329	1.49755	2.7141
2	4.3529	0.86492	0.73972	2.41105
2.5	5.3553	0.82358	0.36484	2.18721
2.9	6.5264	0.79478	0.20179	2.04840
3.0	6.8969	0.78812	0.1728	2.01824
3.1	7.3085	0.78166	0.14747	1.98970
3.5	9.5358	0.75773	0.074308	1.89056
4	14.9716	0.73184	0.025899	1.79723
4.9	171.414	0.69592	0.000175	1.7355

For $n = 1$ and $n = 5$ the integral I_{1202} is analytically derived

$$I_{n=1} = \int_0^{\pi} \frac{\sin^3 x}{x}\, dx = \frac{1}{4}(3\mathrm{Si}(\pi) - \mathrm{Si}(3\pi)) = 0.97025, \quad (35)$$

$$I_{n=5} = \int_0^{\infty} \frac{x^2}{(1+x^3/3)^{7/2}}\, dx = \frac{2\sqrt{3}}{5}. \quad (36)$$

Eq. 34 with the polytropic index $n = 3$ gives the known value $k_{GR} = -0.9183$ its negative sign leads to instability of supermassive stars in GRT.

4. Conclusion

The explicit polytropic equations for integrals providing relativistic corrections to the Newtonian energy ΔU_{GR} by I_{1202} are

$$\int_0^{\xi_1} \vartheta^{n+1}(\vartheta')\xi^3\, d\xi = -\frac{3}{n+2}\int_0^{\xi_1} \vartheta^{n+2}\xi^2\, d\xi,$$

$$\int_0^{\xi_1} \vartheta(\vartheta')^2\xi^2\,d\xi = \frac{1}{2}\int_0^{\xi_1} \vartheta^{n+2}\xi^2\,d\xi,$$

$$\int_0^{\xi_1} (\vartheta')^3\xi^3\,d\xi = \frac{n-10}{2(n+2)}\int_0^{\xi_1} \vartheta^{n+2}\xi^2\,d\xi,$$

$$\int_0^{\xi_1} \vartheta^n(\vartheta')^2\xi^4\,d\xi = -\frac{1}{3}(\vartheta')^3\xi^4 - \frac{(n+1)(n-8)}{3(n+2)}\int_0^{\xi_1} \vartheta^{n+2}\xi^2\,d\xi.$$

Differentiating the functions of $f_{abcd} = \vartheta^{an+b}(\vartheta')^c\xi^d$-type with respect to ξ defines a politropic model. For example, the radial Eq. 10 dependence of the classical polytropic energy is given this way. Eqs. 3 and 3′ could be of particular assistance in defining the Post-Newtonian corrections to the gravitational energy in various relativistic gravitational theories. In GRT this correction ΔE_{GR} is expressed by one integral of I_{abcd} type Eq. 28 whereas it is generally believed that the required number of integrals should be at least two [2–6].

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