GRAVITATION Value 1 Issue 1 1995
Discussing journal on gravitational physics .... *page 36–40*

ENERGY OF GRAVITATIONAL FIELD

ENERGY OF STATIC SPHERICAL–SYMMETRICAL FIELD IN GRAVIDYNAMICS

S.A. Oshepkov

C o n t e n t s

1. Introduction 36
2. Laws of Conservation 37
3. Energy of field 39
4. Conclusion 39
References 40

1. Introduction

Poincare H. [9], Thirring W.E. [11] and Birkhoff G.D. [10] formed an approach to the theory of gravitation. The field theory of gravitation (Gravidynamics) was developed by Baryshev Yu.V. and Sokolov V.V. [1–8]. In GD the gravitational field is described as a tensor field of the second range Ψ^{ik} in the flat space-time of Minkowski. The gauge invariance of the field equations requires the Lagrangian $\mathcal{L}_{(g)}$ of the gravitational field connected with sources as follows

$$\mathcal{L}(g) = -\frac{1}{16\pi G}\left[\mathcal{L}_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\right] \tag{1.1}$$

where

$$\mathcal{L}_1 = 2\Psi_{mn}{}^{;n}\Psi^{lm}{}_{;l}, \qquad \mathcal{L}_2 = -\Psi_{mn;l}\Psi^{mn;l}, \tag{1.2}$$

$$\mathcal{L}_3 = -2\Psi_{mn}{}^{;m}\Psi^{;n}, \qquad \mathcal{L}_4 = \Psi_{;n}\Psi^{;n}. \tag{1.3}$$

The interaction Lagrangian $\mathcal{L}_{(mg)}$ is determined by the principle of the gravitational interaction universality and is postulated as

$$\mathcal{L}(mg) = -\frac{1}{c^2}\,\Psi_{ik}T^{ik}_{(\Sigma)} \tag{1.4}$$

where $T^{ik}_{(\Sigma)}$ is the total Energy Momentum Tensor (EMT) of any material system. The gauge invariance $T^{ik}_{(\Sigma);k} = 0$ follows from the law of conservation of energy-impulse $\mathcal{L}_{(mg)}$. Consequently, in Guilbert–Lorentz' gauge the equations of field

$$\Psi^{ik}{}_{;k} = \frac{1}{2}\,\Psi^{;i} \tag{1.5}$$

take a form

$$\Box(\Psi^{ik} - \frac{1}{2}\eta^{ik}\Psi) = \frac{8\pi G}{c^2} T^{ik}_{(\Sigma)} \tag{1.6}$$

where η^{ik} is the metric tensor in pseudo-Euclid space-time of Minkowski. It should be remarked that Guilbert–Lorentz' gauge provides respectively the law of conservation of the energy-impulse of source [8] through the equation of field (Eq. 16).

The all three classical gravitational experiments [1, 2, 5, 6, 8] are naturally depicted by data of the field equation (1.6) with the corresponding equations of movement. As this takes place, according to Gupta [12] and Thirring [11], the full EMT $T^{ik}_{(\Sigma)}$ includes gravitational field and the field equations (1.6) can be solved by iterations.

2. Laws of Conservation

Laws of conservation, classical EMT, Guilbert's EMT and tensor of spin for the tensor field of the second range Ψ_{ik} can be obtained by the standard rules of Lagrangian formalism. The equation of the field Lagrangian may be written in general form as

$$S = \frac{1}{c}\int\limits_{\Omega} d\Omega\ \mathcal{L}(g_{ik}, \partial_n g_{ik}, g^{ik}, \partial_n g^{ik}, \Psi_{ik}, \partial_n \Psi^{ik}). \tag{2.1}$$

Let us consider an infinitely small change in the coordinates

$$x'^k = x^k + \delta x^k \tag{2.2}$$

where δx^k is the infinitesimal 4-vector. Under change in the coordinates we get

$$\delta S = \frac{1}{c}\int\limits_{\Omega'} d\Omega'\ \mathcal{L}'(x') - \frac{1}{c}\int\limits_{\Omega} d\ \Omega\mathcal{L}(x) = 0 \tag{2.3}$$

by invariance of action, as a scalar and so Jacobian J of the transformation (2.2) is

$$J = 1 + \partial_k \delta x^k, \tag{2.4}$$

then we get

$$\delta S = \frac{1}{c}\int\limits_{\Omega} d\Omega\ [\delta_L \mathcal{L}(x) + \partial_k(\delta x^k \mathcal{L}(x)] \tag{2.5}$$

where $\delta_L \mathcal{L}(x) = \mathcal{L}'(x) - \mathcal{L}(x)$ is Lie's variation. Respectively, it is equal to

$$\begin{aligned} \delta_L \mathcal{L}(x) \quad = \quad & \frac{\partial \mathcal{L}}{\partial g_{ik}}\delta_L g_{ik} + \frac{\partial \mathcal{L}}{\partial g_{ik,n}}\delta_L g_{ik,n} + \frac{\partial \mathcal{L}}{\partial g^{ik}}\delta_L g^{ik} + \frac{\partial \mathcal{L}}{\partial g^{ik}{}_{,n}}\delta_L g^{ik}{}_{,n} + \\ & + \frac{\partial \mathcal{L}}{\partial \Psi^{ik}}\delta_L \Psi^{ik} + \frac{\partial \mathcal{L}}{\partial \Psi^{ik}{}_{,n}}\delta_L \Psi^{ik}{}_{,n} \end{aligned} \tag{2.6}$$

and so $\delta_L g^{ml} = -g^{il}g^{km}\,\delta_L g_{ik}$ then

$$\delta S = \frac{1}{c}\int\limits_{\Omega} d\Omega \left[\frac{\delta \mathcal{L}}{\delta \Psi^{ik}}\delta_L \Psi^{ik} + \left(\frac{\delta \mathcal{L}}{\delta g_{ik}} - g^{il}g^{km}\frac{\delta \mathcal{L}}{\delta g^{lm}}\right)\delta_L g_{ik} + \partial_n J^n\right] \tag{2.7}$$

where

$$J^n = \mathcal{L}\,\delta x^n + \frac{\partial \mathcal{L}}{\partial g_{ik,n}}\delta_L g_{ik} + \frac{\partial \mathcal{L}}{\partial g^{ik}{}_{,n}}\delta_L g^{lm} + \frac{\partial \mathcal{L}}{\partial \Psi^{ik}{}_{,n}}\delta_L \Psi^{ik} \tag{2.8}$$

and $\delta\mathcal{L}/\delta X = \partial\mathcal{L}/\partial X - \partial_n(\partial\mathcal{L}/\partial X_{,n})$ is Euler variation derivative. The density of Lagrangian $\mathcal{L}$ is the scalar density of weight +1, then vector J^n is the 4-vector of weight +1 as well. Then $\partial_n J^n = D_n J^n$. We get by the law of transformation of tensors

$$\delta_L g_{ik} = -g_{il}\partial_k \delta x^l - g_{kl}\partial_i \delta x^l - \delta x^l \partial_l g_{ik} = -g_{il} D_k \delta x^l - g_{kl} D_i \delta x^l \tag{2.9}$$

and

$$\delta_L \Psi_{ik} = -\Psi_{il}\partial_k \delta x^l - \Psi_{kl}\partial_i \delta x^l - \delta x^l \partial_l \Psi_{ik} = -\Psi_{il} D_k \delta x^l - \Psi_{kl} D_i \delta x^l - \delta x^l D_l \Psi_{ik}. \tag{2.10}$$

Substituting (2.9) and (2.10) to (2.7), determining EMT of Guilbert as

$$T^{ik} = -2\left(\frac{\delta\mathcal{L}}{\delta g_{ik}} - g^{il} g^{km} \frac{\delta\mathcal{L}}{\delta g^{lm}}\right) \tag{2.11}$$

and using integration by the parts we get

$$\begin{aligned} \delta S &= \frac{1}{c}\int\limits_{\Omega} d\Omega \left[-\delta x^l \left(D_k T_l^k + \frac{\delta\mathcal{L}}{\delta\Psi^{ik}} D_l \Psi^{ik} + 2D_m\left(\frac{\delta\mathcal{L}}{\delta\Psi^{lk}}\Psi^{mk}\right)\right) + \right. \\ & \left. + D_n\left(J^n + T_l^n \delta x^l + 2\frac{\delta\mathcal{L}}{\delta\Psi^{ik}}\Psi^{in}\delta x^k\right)\right]. \end{aligned} \tag{2.12}$$

Expanding Lie's variation in Eq. 2.8 for vector J^n according to Eqs. 2.9 and 2.10 and combining their terms at δx^l and $D_n \delta x^l$, we have

$$J^n + 2\frac{\delta\mathcal{L}}{\delta\Psi^{ik}}\Psi^{in}\delta x^k = -\tau_l^n \delta x^l - \sigma_l^{nk} D_k \delta x^l \tag{2.13}$$

where

$$\tau_l^n = -\mathcal{L}\delta_l^n + \frac{\delta\mathcal{L}}{\delta\Psi^{ik}_{\ ,n}} D_l \Psi^{ik} - 2\frac{\delta\mathcal{L}}{\delta\Psi^{il}}\Psi^{in} \tag{2.14}$$

is the ordinary EMT and

$$\sigma_l^{nk} = 2\left(\frac{\delta\mathcal{L}}{\partial g_{ik,n}} g_{il} - \frac{\delta\mathcal{L}}{\partial g^{li}_{\ ,n}} g^{ki} - \frac{\delta\mathcal{L}}{\partial\Psi^{il}_{\ ,n}}\Psi^{ik}\right) \tag{2.15}$$

is the tensor of spin. Expanding the covariant divergence in Eq. 2.12 we finally get

$$\begin{aligned} \delta S &= \frac{1}{c}\int\limits_{\Omega} d\Omega \left[-\delta x^l \left(D_n \tau_l^n + \frac{\delta\mathcal{L}}{\delta\Psi^{ik}} D_l \Psi^{ik} + 2D_m\left(\frac{\delta\mathcal{L}}{\delta\Psi^{lk}}\Psi^{mk}\right)\right) + \right. \\ & \left. + D_n \delta x^l \left[-\tau_l^n + T_l^n - D_k \sigma_l^{kn}\right] - \sigma_l^{nk} D_n D_k \delta x^l\right] = 0. \end{aligned} \tag{2.16}$$

at unrestricted δx^l and the desired strong laws of conservation are

$$D_n \tau_l^n + \frac{\delta\mathcal{L}}{\delta\Psi^{ik}} D_l \Psi^{ik} + 2D_m\left(\frac{\delta\mathcal{L}}{\delta\Psi^{lk}}\Psi^{mk}\right) = 0, \tag{2.17}$$

$$T_l^n - \tau_l^m - D_k \sigma_l^{kn} = 0. \tag{2.18}$$

The equations of field $\delta\mathcal{L}/\delta\Psi^{ik} = 0$ lead to the weak laws of the conservation of energy and impulse

$$D_n \tau_l^n = D_n T_l^n = 0, \tag{2.19}$$

$$T_l^n - \tau_l^m = D_k \sigma_l^{kn}. \tag{2.20}$$

It should be noted that these weak laws will be fulfilled in both cases: 1) at the free field $\mathcal{L} = \mathcal{L}_{(g)}$ or 2) at the dependent field $\mathcal{L} = \mathcal{L}_{(g)} + \mathcal{L}_{(mg)}$ but off a body when τ_l^n and T_l^n are the pure gravitational EMT of field.

3. Energy of Field

Canonical EMT of Field.

The Lagrangian $\mathcal{L}_{(g)}$ gives the canonical EMT of field by Eq. 2.14 that has a form [1]

$$\tau^{ik}_{(g)} = \frac{1}{8\pi G}\left(\Psi^{lm;i}\Psi_{lm}{}^{;k} - \frac{1}{2}\eta^{ik}\Psi_{lm;n}\Psi^{lm;n} - \frac{1}{2}\Psi^{;i}\Psi^{;k} + \frac{1}{4}\eta^{ik}\Psi_{;l}\Psi^{;l}\right). \qquad (3.1)$$

It is a characteristic feature of Gravidynamics that this canonical EMT is a symmetrical one.

At a first approximation, the field of Eq. 1.6 leads to the gravitational field of the static spherical symmetrical case when $T^{ik}_{(\Sigma)} = T^{ik}_{(m)} = mc^2\delta(\mathbf{r})$ received from Eq. 1.6

$$\Psi^{ik} = \varphi_{(N)}\delta^{ik} \qquad (3.2)$$

where $\varphi_{(N)} = -GM/R$ is Newton's potential. This potential was pioneered by G. Birkhoff [10]. Birkhoff's potential gives the corresponding energy of gravitational field [1]

$$\tau^{oo}_{(g)} = +\frac{1}{8\pi G}\,(\nabla\varphi_{(N)})^2. \qquad (3.3)$$

It should be noted that the gain energy is not only of positive and its dependence of potential is an analogue to the corresponding one for vector field in Electrodynamics but it can be experimentally tested within Gravidynamics for the energy of gravitational field, being the source of the right-hand part of the field equation, to a second approximation of the theory, gives the deficit $7''$ in a century for perigelius shift of Mercury, that is, about 16 % of the whole effect.

EMT of Guilbert.

Energy Momentum Tensor of Guilbert (2.11) for $\mathcal{L} = \sqrt{-g}\mathcal{L}_{(g)}$ where $\mathcal{L}_{(g)}$ is defined according to Eq. 1.1 and at Guilbert–Lorentz' gauge in the pseydo-Euclidian coordinates of Minkowski's space-time has a form as

$$T^{pq}_{(g)} = -\frac{1}{16\pi G}\,[\Psi^{,q}\Psi^{,p} - \frac{1}{2}\eta^{pq}\Psi_{,n}\Psi^{,n} + \eta^{pq}\Psi_{lm,n}\Psi^{lm,n} - 2\Psi_{lm}{}^{,p}\Psi^{lm,q} + 2\Psi^{,p}_{,n}\Psi^{qn} +$$
$$+2\Psi^{,p}_{,n}\Psi^{qn} - +2\Psi^{,n}_{,n}\Psi^{pq} - 4\Psi^{p,q}_{m,n}\Psi^{qm} - 4\Psi^{p,q}_{m,n}\Psi^{nm} - 2\Psi^{p,q}_{m}\Psi^{,m} + 4\Psi^{n,q}_{m}\Psi^{pm}_{,n}]. \qquad (3.4)$$

For Birkhoff's potential (3.2), that is, for the static spherical-symmetrical case off body we get the energy density as

$$T^{oo}_{(g)} = +\frac{1}{8\pi G}\,(\nabla\varphi_{(N)})^2 \qquad (3.5)$$

that agrees to calculation of τ^{oo} (3.3). By Eq. 2.20 this defines the divergence of spin tensor of σ^{ko}_{o}-component as zero as well as in Electrodynamics.

4. Conclusion

In GD the energy density of gravitational field ε of the static spherical symmetry case determined by the standard and Guilbert EMT is

$$\varepsilon = +\frac{1}{8\pi G}\,(\nabla\varphi_{(N)})^2. \qquad (4.1)$$

The pseudo-tensor of the energy-impulse of GR defined by Landau gives

$$T^{oo}_{(GR)} = -\frac{7}{8\pi G}\left(\nabla\varphi_{(N)}\right)^2 \qquad (4.2)$$

for the energy density of the said case. According to Thirring W.E. [11], Cavalleri G. and Spinelli G. [13] the convergence of field approach to GR with the field Lagrangian that is different from $\mathcal{L}_{(g)}$ (1.1) by the divergence addition, that is,

$$\mathcal{L}'_2 = 2\Psi_{mn}{}^{;l}\Psi^{ln}{}_{;m} \qquad (4.3)$$

is under consideration at the place of $\mathcal{L}_2$ (1.2) where $\mathcal{L}'_2$ is the fifth and the last invariant composed of the first derivatives of field potentials. Calculation for this field Lagrangian $\mathcal{L}'_{(g)} = -\frac{1}{16\pi G}\left[\mathcal{L}'_1 + \mathcal{L}_2 + \mathcal{L}_3 + \mathcal{L}_4\right]$ actually gives

$$T'^{oo} = -\frac{7}{8\pi G}(\nabla\varphi_{(N)})^2. \qquad (4.4)$$

Then the divergent correction to the field Lagrangian changes Guilbert's EMT.

The differences between GD and GR at the first order of the theory of the field energy density as well as different views on the existence of the scalar gravitational fields [3] and the stability of supermassive stars [8] allow to state that the field theory of gravitation (Gravidynamics) based on the field Lagrangian $\mathcal{L}_{(g)}$ (1.1) leads to a different, other than GR, nonlinear relativistic theory of gravitational field.

The author would like to express his gratitude to Yu.V. Baryshev and S.N. Sokolov for their discussion of the present paper.

REFERENCES

1. *Sokolov V.V., Baryshev Yu.V.* Field theory approach to gravitation. Energy-momentum tensor of the field. // Gravitation and Relativity Theory, Kazan State Univ., *1980*. (in Russian).
2. *Baryshev Yu.V., Sokolov V.V.* Relativistic Tensor Theory of Gravitational Field in Flat Space-Time. // Trudy Astr. Obsr. LGU, vol. 38, pp. 36–61, *1983*. (in Russian).
3. *Baryshev Yu.V.* To the gravitational radiation of double system with pulsar PSR 1913+16. // Astrofisika, vol. 18, iss. 1, pp. 93–99, *1982*. (in Russian).
4. Baryshev Yu.V., Sokolov V.V. Some astrophysical applications of the dynamical interpretation of gravitation. // Astrofisika, vol. 21, iss. 2, pp. 361–366, *1984*. (in Russian).
5. Baryshev Yu.V. Equations of motion of test particles in Lorentz-covariant theory of gravitation. // Vestnik LGU, ser. 1, iss. 4, pp. 113–118, *1986*. (in Russian).
6. *Baryshev Yu.V.* Conservation laws in equations of motion in field theory of gravitation. // Vestnik LGU, ser. 1, iss. 2, pp. 80–85, *1988*. (in Russian).
7. *Sokolov V.V.* Linear and nonlinear gravidynamics: static fields of a collapsar. // Astroph. and Space Sci., vol. 191, pp. 231–258, *1992*.
8. *Baryshev Yu.V.* Theory of gravitational field: foudations and astrophysical applications. (in print).
9. *Poincare H.* // Bull. des Science Math., vol. 28, ser. 2, pp. 302–328, *1904*.
10. *Birkhgoff G.D.* Flat space-time and gravitation. // Proceed. Nat. Acad. Sci., vol. 30, No. 10, pp. 324–334, *1944*.
11. *Thirring W.E.* An alternative approach to the theory of gravitation. // Ann. of Phys., vol. 16, pp. 96–117, *1961*.
12. *Gupta S.* // Rev. Mod. Phys., vol. 29, p. 334, *1957*.
13. *Cavalleri G. and Spinelli G.* Field-Theoretic Approach to Gravity in the Flat Space-Time. // Rivista del Nuovo Cimento, vol. 3, No. 8, pp. 1–92, *1980*.